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DIFFERENTIAL EQUATIONS

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PART I

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ANNAPOLIS
U. S. Naval Academy Print
1883.

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DIFFERENTIAL EQUATIONS.

I.

Particular and Complete Integrals.

1. An equation involving one or more of the derivatives of a function y, with reference to an independent variable x, and one or both of the variables themselves is called a *differential equation*. The *order* of the equation is the same as that of the highest derivative which enters it, and its *degree* is indicated by the exponent of the highest power of this derivative which occurs. Thus

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \tag{1}$$

is a differential equation of the second order and of the first degree.

It is readily shown that

$$y = e^x \sin x \tag{2}$$

satisfies this equation; but this is not the general solution of the equation, for $y = e^x \cos x$ is also a solution, and in fact the most general solution is

$$y = A\,e^x \sin x + B\,e^x \cos x, \tag{3}$$

where A and B are any constant quantities.

2. The simplest type of differential equations is

$$\frac{dy}{dx} = f(x),$$

or

$$dy = f(x)\,dx;$$

whence by integration

$$y = \int f(x)\,dx.$$

If we regard the integral in this equation as a particular value of the indefinite integral, the most general solution of equation (1) is expressed by the complete integral

$$y = \int f(x)\,dx + C. \tag{3}$$

In like manner, a solution of any differential equation is called a *particular integral* of the equation, and the most general solution is called the *complete integral.* Thus, in Art. 1, equation (2) is a particular integral and (3) is the complete integral of equation (1).

The constants which occur in the complete integral and not in the differential equation are called *constants of integration.*

3. The solution of a differential equation may be in the form of a relation between x and y. In this case, the verification consists in showing that the given differential equation can be derived from the integral equation either directly by differentiation or by combining it with the equations resulting from its differentiation. For example, the solution of

$$(x^2 - y^2)\frac{dy}{dx} - 2xy = 0 \tag{1}$$

is

$$x^2 + y^2 - cy = 0; \tag{2}$$

for, if we differentiate this equation, we have

$$2x + (2y - c)\frac{dy}{dx} = 0, \tag{3}$$

and the result of eliminating c between equations (2) and (3) is found to be equation (1).

4. An equation containing x, y, and one or more arbitrary constants from which a differential equation independent of these constants is derived is sometimes called the *primitive* of the differential equation. Thus, equation (2) is the primitive of equation (1). When the primitive contains a single arbitrary constant, a differential equation of the first order can be obtained, the constant either disappearing on differentiation, or as in the example, being eliminated from the direct result of differentiation by means of the primitive equation itself. In like manner, if the primitive contains two arbitrary constants, we proceed to a second differentiation and eliminate the two constants between the primitive and the two differential equations, the result being a differential equation of the second order. It is thus evident that in the complete integral of an equation of the first order but one arbitrary constant can occur, as in equation (2) of the preceding article, while two and two only must occur in the complete integral of a differential equation of the second order, and so on.

Separation of the Variables.

5. A differential equation of the first order and degree may be written in the form

$$M\,dx + N\,dy = 0,$$

M and *N* being in general functions of *x* and *y*. The simplest case is that in which *M* is a function of *x* only, and *N* a function of *y* only, in other words when the equation can be written in the form

$$f(x)\,dx + \phi(y)\,dy = 0, \qquad (1)$$

of which the complete integral is evidently

$$\int f(x)\,dx + \int \phi(y)\,dy = C. \qquad (2)$$

6. The process of reducing an equation, when possible, to the form (1) is called *the separation of the variables*. Thus, in the equation

$$(1-y)\,dx + (1+x)\,dy = 0 \qquad (1)$$

the variables are seperated by dividing by $(1+x)\,(1-y)$; whence

$$\frac{dx}{1+x} + \frac{dy}{1-y} = 0, \qquad (2)$$

and integrating

$$\log(1+x) - \log(1-y) = C. \qquad (3)$$

The solution here presents itself in a transcendental form, but it is readily reduced to an algebraic form; thus form (3)

$$\log\frac{1+x}{1-y} = C$$

whence

$$\frac{1+x}{1-y} = e = C,$$

or

$$1+x = C(1-y),$$

in which C is put for the constant e.

Homogenious Equations of the First Order and Degree.

7. The equation

$$M\,dx + N\,dy = 0$$

is said to be *homogeneous* when *M* and *N* are homogeneous functions of *x* and *y* of the same degree. Since the ratio of two homogeneous functions of the same degree is a function of $\frac{y}{x}$ we may write the equation in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right);$$

if then we put $y = xz$, whence $\frac{dy}{dx} = x\frac{dz}{dx} + z$, we have

$$x\frac{dz}{dx} + z = f(z),$$

in which the variables can be separated, thus,

$$\frac{dx}{x} = \frac{dz}{f(z) - z}.$$

For example equation (1) of Art. 3,

$$(x^2 - y^2)\frac{dy}{dx} - 2xy = 0$$

is homogeneous. Putting $y = xz$ we obtain

$$x\frac{dz}{dx} + z = \frac{2z}{1 - z^2}$$

whence $$x\frac{dz}{dx} = \frac{z + z^3}{1 - z^2}$$

or $$\frac{dx}{x} = \frac{1 - z^2}{z(1 + z^2)}dz$$

Decomposing the fraction

$$\frac{dx}{x} = \frac{dz}{z} - \frac{2z\,dz}{1 + z^2};$$

therefore $$\log x = \log\frac{z}{1 + z^2} + C,$$

and replacing z by $\frac{y}{x}$,

$$\log\frac{x^2 + y^2}{y} = C$$

or $$x^2 + y^2 = Cy.$$

8. The equation $Mdx + Ndy = 0$ can always be solved when M N are functions of the first degree in x and y, that is when the equation is of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}; \tag{1}$$

for substitute in this

$$x = \xi + h, \qquad y = \eta + k,$$

and we have

$$\frac{d\eta}{d\xi} = \frac{a\xi + b\eta + ah + bk + c}{a'\xi + b'\eta + a'h + b'k + c'}. \tag{2}$$

If now we determine h and k by the equations

$$\left.\begin{aligned} ah + bk + c &= 0 \\ a'h + b'k + c' &= 0 \end{aligned}\right\}, \tag{3}$$

equation (2) takes the homogeneous form

$$\frac{d\eta}{d\xi} = \frac{a\xi + b\eta}{a'\xi + b'\eta}.$$

from which we can determine the integral equation between ξ and η, and thence that involving x and y.

Equations (3) give impossible values of h and k

when $$\frac{a'}{a} = \frac{b'}{b} = m,$$

In this case (1) can be written in the form

$$\frac{dy}{dx} = \frac{ax + by + c}{m\,(ax + by) + c'};$$

and putting $ax + by = z$, whence

$$a + b\frac{dy}{dx} = \frac{dz}{dx},$$

we have $$\frac{dz}{dx} - a = b\frac{z + c}{mz + c'}$$

in which the variables can be separated.

Exact Differential Equations.

9. An expression which is the direct result of differentation is called *an exact differential*, and an equation in which an exact differential is put equal to zero is called *an exact differential equation*. Thus, if u is a function of x and y

$$du = 0$$

is an exact differential equation; and its complete integral is

$$u = C.$$

The exact differential du is the form

$$du = Mdx + Ndy$$

in which M and N are the partial derivatives of u; that is.

$$M = \frac{du}{dx} \qquad N = \frac{du}{dy}:$$

hence, as shown in Art. 403, Diff. Calc., we must have.

$$\frac{dM}{dy} = \frac{dN}{dx}, \tag{1}$$

each of the members of this equation being in fact a value of $\dfrac{d^2u}{dx\,dy}$.

Therefore (1) is a necessary condition in order that

$$M\,dx + N\,dy = 0 \tag{2}$$

should be an exact differential equation.

For example the equation

$$2y\,dx + x\,dy = 0$$

is *not* exact, because $M = 2y$ and $N = x$, so that (1) is not satisfied; but if we multiply the equation by x, we have

$$2\,xydx + x^2dy = 0$$

which is exact because $M = 2xy$, $N = x^2$, so that

$$\frac{dM}{dy} = 2x = \frac{dN}{dx}.$$

8. When the condition (1) is satisfied, the complete integral $u = C$ is found by direct integration. Since

$$du = M\,dx + N\,dy,$$

the differential of u taken on the supposition that y is constant is $M\,dx$, hence

$$u = \int M dx + Y \tag{1}$$

where Y, which takes the place of the constant of integration, is independent of x, but may involve y. To find Y, we have from (1), since $\dfrac{du}{dy} = N$

$$N = \frac{d}{dy}\int M\,dx + \frac{dY}{dy}$$

or

$$\frac{dY}{dy} = N - \frac{d}{dy}\int M\,dx, \tag{2}$$

from which Y is to be found by integration. Thus, in the example given in Art. 7, $M = 2xy$ and (1) gives

$$u = x^2y + Y,$$

and (2) gives, since $N = x^2$,

$$\frac{dY}{dy} = N - x^2 = 0;$$

therefore $$Y = C$$

and the complete integral is

$$x^2 y + C = 0.$$

9. It is to be noticed that the second member of equation (2) will be found independent of x, for its derivative with respect to x is

$$\frac{dN}{dx} - \frac{dM}{dy},$$

which is zero by the condition for an exact differential. The expression in fact denotes those terms of N which are independent of x, It is evident that the integrals of these terms, and also of those containing x only, may be considered separately, provided the terms containing both x and y constitute an exact differential. It is therefore only necessary to apply the criterion to these terms. Thus, in the equation

$$(2x + y)\,dx + (x - y)\,dy = 0,$$

these terms are $y\,dx + x\,dy$, which is the differential of xy, hence the integral is

$$x^2 + xy - \tfrac{1}{2} y^2 = C.$$

10. An expression involving only some function of x and y and the differential of this function is obviously an exact differential. Thus, in the equation

$$\frac{x\,dx + y\,dy}{\sqrt{(x^2 + y^2 + 1)}} + \frac{y\,dx - x\,dy}{x^2 + y^2} = 0,$$

the first term is a function of $x^2 + y^2$ and its differential, and the second term is a function of $\frac{x}{y}$ and its differential. In other words, the equation can be written in the form

$$\frac{d(x^2 + y^2)}{2\sqrt{(x^2 + y^2 + 1)}} + \frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y}\right)^2 + 1} = 0$$

hence the integral is

$$\sqrt{(x^2 + y^2 + 1)} + \tan^{-1}\frac{x}{y} = C.$$

Integrating Factors.

11. As shown in Art. 4, a differential equation of the form

$$M\,dx + N\,dy = 0 \qquad (1)$$

may be regarded as derived from a primitive containing an arbitrary constant C by the elimination of C between the primitive and the equation derived from it directly by differentation. If however the primitive be reduced to the form

$$f(x, y) = C, \qquad (2)$$

the constant will disappear by direct integration, and the result must be equivalent to equation (1). We infer therefore that the differential of equation (2) is of the form

$$\mu\,(M\,dx + N\,dy) = 0, \qquad (3)$$

in which μ is a factor which generally contains x and y.

Equation (3) is an exact differential equation, and the factor μ which converts (1) into an exact differential equation is called an *integrating factor*.

A differential equation of the form (1) has a variety of integrating factors corresponding to different forms of the constant of integration. For example, if we differentiate equations (3) and (4) of Art. 6, we obtain equation (1) multiplied respectively by

$$\frac{1}{(1+x)(1-y)}, \quad \text{and} \quad \frac{1}{(1-y)^2};$$

each of these expressions is therefore an integrating factor of equation (1).

12. An integrating factor is sometimes suggested by the form of the differential equation. For example in the equation

$$(y + \log x)\,dx - x\,dy = 0$$

the terms containing both x and y are

$$y\,dx - x\,dy$$

which become an exact differential when divided by y^2 or by x^2, and since the remaining term contains x only, $\frac{1}{x^2}$ is an integrating factor. Thus we write

$$\frac{y\,dx - x\,dy}{x^2} + \frac{\log x\,dx}{x^2} = 0$$

whence integrating

$$-\frac{y}{x} - \frac{\log x}{x} - \frac{1}{x} + C = 0,$$

or $$Cx + y + \log x + 1 = 0.$$

The Linear Equation of the First Order.

13. A differential equation is said to be *linear* when it is of the first degree with respect to y and its derivatives. The linear equation of the first order may therefore be written in the form

$$\frac{dy}{dx} + Py = Q,$$

in which P and Q may be any functions of x.

If the coefficient of y is constant, the form is

$$dy + a\,y\,dx = Q\,dx \tag{1}$$

of which e^{ax} is an integrating factor. Thus, we have

$$e^{ax}\,dy + a\,e^{ax}y\,dx = e^{ax}\,Qdx,$$

and the complete integral is

$$e^{ax}y = \int e^{ax}Qdx + C. \tag{2}$$

For example, the integral of

$$\frac{dy}{dx} - y = e^{nx}$$

is $$e^{-x}y = \int e^{(n-1)x}\,dx + C$$

or $$y = \frac{e^{nx}}{n-1} + Ce^{x}.$$

14. The linear equation with a variable coefficient

$$\frac{dy}{dx} + Py = Q \tag{1}$$

is solved in a similar manner, the integrating factor being $e^{\int Pdx}$. Thus,

$$e^{\int Pdx}\,dy + e^{\int Pdx}\,Pydx = e^{\int Pdx}\,Qdx,$$

whence integrating

$$e^{\int Pdx}y = \int e^{\int Pdx}\,Qdx + C. \tag{2}$$

It is unnecessary to add an arbitrary constant to the integral $\int Pdx$ since the presence of a single arbitrary constant is sufficient in the solution of an equation of the first order: and in fact if we put $\int Pdx + c$ in place of $\int Pdx$ in equation (2) we have

$$e^c e^{\int Pdx} y = e^c \int e^{\int Pdx} Qdx + C^1$$

which is plainly equivalent to (2) if $C^1 = e^c C$.

15. It is of course impossible to express the complete integral of the general equation (1) without the aid of the integral sign, but in some cases it is possible to evaluate the integrals in equation (2). Thus, in the linear equation

$$\frac{dy}{dx} + \frac{y}{x+1} = x - 1,$$

we have $P = \frac{1}{x+1}$, whence

$$\int Pdx = \log(x+1), \quad e^{\int Pdx} = x + 1,$$

and equation (2) gives

$$(x+1)y = \int (x^2 - 1)\, dx$$

or

$$y = \frac{x^3 - 3x + C}{3(x+1)}.$$

Extension of the Linear Equation.

16. An equation of the form

$$\frac{dy}{dx} + Py = Qy^n,$$

where P and Q are functions of x, can be reduced to the linear form; for dividing by y^n we have

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q,$$

and putting

$$z = y^{1-n} \qquad \text{whence} \qquad dz = (1-n)\, y^{-n}\, dy,$$

this becomes

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q.$$

If we change the sign of n, the equation takes the form

$$y^n \frac{dy}{dx} + Py^{n+1} = Q,$$

which is accordingly reduced to a linear form by putting $z = y^{n+1}$.

II.

Equations of the First Order, but not of the First Degree.

17. Let $$p = \frac{dy}{dx};$$

then an equation of the first order is a relation between x, y, and p, which should, if possible, be solved for p.

Suppose the equation to be algebraic, and, for example, of the second degree with respect to p. If the two values found for p are rational functions of x and y, the equation is decomposable into two equations of the first degree. Thus,

$$\left(\frac{dy}{dx}\right)^2 - a^2y^2 = 0 \qquad (1)$$

is equivalent to the two equations

$$\frac{dy}{dx} - ay = 0,$$

and $$\frac{dy}{dx} + ay = 0,$$

whose solutions are

$$y = Ce^{ax},$$

and $$y = Ce^{-ax},$$

either of which is a solution of equation (1).

18. In a proper, that is, an *indecomposable* equation of the first order and second degree, the two expressions for p are the values of a two-valued function of x and y. Thus, the equation

$$\left(\frac{dy}{dx}\right)^2 = \frac{1-y^2}{1-x^2} \qquad (1)$$

gives $$\frac{dx}{\sqrt{(1-x)^2}} \pm \frac{dy}{\sqrt{(1-y)^2}} = 0.$$

Integrating, we have

$$\sin^{-1} x + \sin^{-1} y = C, \qquad (2)$$

in which the multiple values of $\sin^{-1} y$ render it unnecessary to retain the ambiguous sign. This result may be reduced to an algebraic form: for, taking the cosine of each member, we have

$$\sqrt{1-x^2}\,\sqrt{1-y^2} - xy = \cos C = c,$$

and rationalizing,

$$1 - x^2 - y^2 = c^2 + 2cxy, \qquad (3)$$

an equation of the second degree with respect to the constant of integration c.

Geometrical Meaning of a Differential Equation of the First Order.

19. If in a differential equation of the first order x and y be taken as the rectangular coordinates of a point, we have

$$p = \frac{dy}{dx} = \tan \phi,$$

where ϕ is the inclination to the axis of x of the direction in which the point is moving. We may assume any values for x and y, that is, any position for the point, the equation then determines the value of p, and therefore the direction in which the point moves. If the point moves in the assigned direction from its initial position, the value of p will in general vary, and the point will move in a curve whose equation is an integral of the differential equation. The complete integral is the equation of a system of curves determined in this manner, c being an arbitrary parameter, whose value depends upon the position of the assumed initial point.

20. If the equation is of the first degree, the value of p is real for every position of the assumed point, and there is but one value of p corresponding to a given point. Hence, the system of curves is such that one curve and only one passes through a given point. In other words, no two curves of the system intersect; and, as the parameter c passes through all its possible values, the variable curve sweeps over the whole plane. Accordingly, if the coordinates of the assumed point are substituted in the complete integral, the resulting equation determines a single real value of c. Hence the complete integral may be expressed as an equation of the first degree with respect to c.

21. If the differential equation is of the second degree, there will be, for each position of the assumed point, two real or imaginary values of p. Hence, the system of curves represented by the completeintegral is, in this case, such that two curves pass through every point for which the values of p are real. Hence, if the coordinates of such a point be substituted in the complete integral, the resulting equation determines two values of c: and we infer that the complete integral may be expressed as an equation of the second degree with respect to c.

In like manner, it may be shown that *when the differential equation is of the* n*th degree with respect to* p, *the integral equation is of the* n*th degree with respect to* c.

Singular Solutions.

22. A differential equation of the first order and second degree may be written in the form

$$Lp^2 + Mp + N = 0, \tag{1}$$

in which L, M and N are in general functions of x and y, and, if P be the assumed point, they will have definite numerical values for each position of P. The values of p will generally be real when P is in a certain part of the plane, and imaginary for the rest of the plane; while, on the boundary separating these regions, the values of p will be equal. The condition that (1) shall have equal roots, is

$$M^2 - 4LN = 0. \tag{2}$$

This is an equation between x and y which must be satisfied when P is on the boundary line.

23. If now the system of curves represented by the complete integral has an envelop, it is evident that this envelop will constitute a portion at least of the boundary line; and, since the envelop touches at P, one of the curves of the system, the value of $\frac{dy}{dx}$ for a point moving in the envelop will be identical with the value of p derived from the differential equation. In other words, the equation of the envelop is a solution of the given differential equation. Such a solution, not being included in the complete integral, is called a ***singular solution.***

For example, equation (1) of Art. 18 is

$$(1-x^2)p^2 - (1-y^2) = 0,$$

and the condition for equal roots, is

$$(1-x^2)(1-y^2) = 0,$$

the locus of which consists of the straight lines

$$x = \pm 1, \qquad y = \pm 1.$$

Each of these is found to satisfy the given differential equation, and is a singular solution; the lines are, in fact, the envelops of the system of ellipses and hyperbolas represented by the complete primitive, [equation (3), Art. 18,]

$$x^2 + y^2 + 2cxy + c^2 - 1 = 0.$$

Cusp-Loci and Tac-Loci.

24. It must not be inferred that $M^2 - 4LN = 0$, always represents a singular solution, although it includes the singular solution if there be one. For, in the first place, suppose that the curve represented by the complete integral has a cusp for all values of c; it is obvious that the values of p will be real upon one side of the line described by the cusp as c varies, and imaginary upon the other side; while, the values of p for points on this line, which is called a *cusp-locus*, will be equal, but *not* identical with the value of $\frac{dy}{dx}$ for a point moving upon the cusp-locus. Hence, if there be a cusp-locus, it will be included in the locus of

$$M^2 - 4LN = 0,$$

but will *not* satisfy the given differential equation.

For example, the equation

$$4p^2 = 9x$$

has equal roots when

$$x = 0,$$

but this does not satisfy the differential equation. The complete integral in this case is

$$(y + c)^2 = x^3,$$

which represents a system of semi-cubical parabolas whose cusps are situated upon the axis of y, which is thus a cusp-locus.

25. In the next place, suppose that the curve touches another curve of the system; then the two values of p are equal at the point of contact; as c varies, this point describes a line called a *tac-locus*, for every point of which the values of p are equal; the equation of this line is therefore included in the equation

$$M^2 - 4LN = 0;$$

but it will not satisfy the differential equation.

It is to be noticed that, since the values of p are real upon both sides of a tac locus, the expression M^2-4LN does not change sign as we cross a tac locus. For this reason, *the factor which vanishes at a tac-locus generally appears as a square factor, and in all cases with an even exponent.* As an example let us take the equation

$$4xp^2-(3x-a)^2=0. \tag{1}$$

The condition for equal roots is

$$x(3x-a)^2=0; \tag{2}$$

$x=0$ satisfies the differential equation and is therefore a singular solution, but the expression $3x-a)^2$ does not change sign as we cross the line

$$x=\frac{a}{3},$$

which, accordingly, is the equation of a tac locus.

Derivation of the Singular Solution from the Complete Integral.

26. It is evident from Art. 22 that when the coordinates of the assumed point are substituted in the complete integral, the values of c, determined from the resulting equation, are real and imaginary respectively in the same portions of the plane in which those of p are real and imaginary. By reasoning similar to that employed in Arts. 22 and 23, it is shown that the condition that c should have equal roots must be satisfied by a singular solution. Supposing the differential equation to be of the second degree, the complete integral is of the form

$$Pc^2+Qc+R=0.$$

where P, Q and R are functions of x and y, and the condition for equal roots is

$$Q^2-4PR=0.$$

Thus, in the example given in Art. 23, this equation is

$$x^2y^2-x^2-y^2+1=0,$$

which is identical with the condition that p shall have equal roots.

Node-loci and Cusp-loci.

27. The equation $Q^2-4PR=0$, like $M^2-4LN=0$, includes the equation of a cusp locus, and for the same reason (compare Art. 24); but it does *not* include the equation of a tac-locus, for the two curves which touch one another in a point of the tac-locus correspond to different values of c.

On the other hand the expression Q^2-4PR may contain a factor which is not contained in $M^2 - 4LN$. For, supposing the curve representing the integral to have a node, p will at the node have two distinct values; but, since these values correspond, not to two distinct curves of the system, but to the two branches of a single curve, there will at this point be but one value of c. Hence, at a node, the condition that, c has equal roots is satisfied, but *not* the condition that p has equal roots.

28. If the curve has a node for all values of c, then, as b varies, the node will describe a line called a *node-locus*, every point of which satisfies the equation

$$Q^2 - 4PR = 0,$$

and whose equation is therefore a factor of this equation. But, since the values of c are real on both sides of the *node-locus*, the expression $Q^2 - 4PR$ does not change sign as the point (x, y) crosses this line; hence *the factor indicating a node-locus generally appears as a square factor, and in all cases with an even exponent.*

For example, the integral of the equation

$$4xp^2 - (3x - a)^2 = 0, \tag{1}$$

or

$$p = \frac{3x - a}{2\sqrt{x}}, \tag{2}$$

given in Art. 25, is

$$y + c = x^{\frac{3}{2}} - ax^{\frac{1}{2}},$$

or

$$(y + c)^2 = x(x - a)^2. \tag{3}$$

The condition that c shall have equal roots is

$$x(x - a)^2 = 0; \tag{4}$$

$x = 0$ is the singular solution as found in Art. 25, and $x = a$ is the locus of the cusp $(a, -c)$ of the curve (3).

29. On the other hand, the values of c being real on one side of an envelop or of a cusp-locus and imaginary on the other, the factor representing either of these curves must appear with an odd exponent; but, *in the case of a cusp-locus, the exponent cannot be less than three.* Thus, if, in the example given in the preceding article, we put $a = 0$, the complete integral (3) becomes

$$(y + c)^2 = x^3$$

representing a system of semi-cubical parabolas, as in Art. 24. The node-locus $x = a$ has now united with the envelop $x = 0$ to form a cusp-locus, and the factors representing them in equation (4) now constitute the triple factor $x^3 = 0$.

30. It would at first sight appear that we should, in like manner, find the cusp-locus represented by a triple factor in the equation $M^2-4LN=0$ derived from the differential equation; thus in the present example, when $a=0$ the tac locus $x=\frac{1}{3}a$, found in Art. 25, unites with the envelop $x=0$. It is however to be noticed that, when $a=0$ in equation (1), this factor divides out, so that when $a=0$, equation (2) of Art. 25 no longer represents the condition for equal roots, which is simply $x=0$, the square of the factor in question having disappeared from the expression M^2-4LN. That the factor must always thus divide out is evident from the fact that the values of p at the tac-locus and at the envelop *do not become identical*, when these lines are brought into coincidence; whence the general value of p [expressed in the present example by equation (2), Art. 28,] must become indeterminate in the special case, and this can only happen when the vanishing factor divides out from the differential equation.

The reasoning by which the results obtained in the preceding articles are established, although, for simplicity, stated with reference to differential equations of the second degree, is evidently applicable also to equations of higher degrees.

Differentiation of a Differential Equation.

31. The differentiation of a differential equation of the first order produces a differential equation of the second order; but, if the given equation can be solved for y, that is, put in the form

$$y=f(x,p),$$

the result will be of the form

$$p=\phi\left(x, p, \frac{dp}{dx}\right),$$

which, not containing y, may be regarded as a differential equation of the first order between the variables x and p. If we can integrate this last equation, we shall have a relation between x, p and a constant of integration, c; but the original equation is a relation between x, y and p; hence, if we eliminate p between these equations we shall have the relation between x y and c, that is, the complete integral.

32. For example, given

$$\frac{dy}{dx}+2xy=x^2+y^2,$$

solving for y, $$y=x+\sqrt{p}, \tag{1}$$

and differentiating,

$$p=1+\frac{1}{2\sqrt{p}}\frac{dp}{dx}.$$

Separating the variables, we have

$$dx=\frac{dp}{2\sqrt{p}\,(p-1)},$$

and integrating,

$$x=\frac{1}{2}\log\frac{\sqrt{p}-1}{\sqrt{p}+1}+c,$$

or $$\frac{\sqrt{p}-1}{\sqrt{p}+1}=e^{2\,(x-c)};$$

whence $$\sqrt{p}=\frac{1+e^{2x-2c}}{1-e^{2x-2c}},$$

and elininating p by means of equation (1),

$$y-x=\frac{e^{2c}+e^{2x}}{e^{2c}-e^{2x}}.$$

33. A differential equation of the first order between p and x may also be obtained, when the given equation is not in the form $y=f\,(x,p)$, by differentiating and eliminating y from the result by means of the original equation; but the result thus found is not usually simpler than the given equation. It is moreover obvious that an equation in the form $x=\phi\,(y,q)$, where q is put for $\frac{dx}{dy}$, can be treated in like manner by differentiating with reference to y,

Some of the cases in which the derived equation is integrable are illustrated in the following articles.

Homogeneous Equations of the First Order.

34. When a homogeneous equation of the first order but not of the first degree cannot be solved for p, it may be possible to solve it for $\frac{y}{x}$; that is, to put it in the form

$$\frac{y}{x}=\phi\,(p),$$

whence $$y=x\,\phi\,(p).$$

Differentiating, $$p = x\phi'(p)\frac{dp}{dx} + \phi(p),$$

in which the variables can be separated, thus

$$\frac{dx}{x} = \frac{\phi'(p)\,dp}{p - \varphi(p)}.$$

Equations from which one of the Variables is Absent.

35. If the equation does not contain x, it should, if possible, be solved for p: it then takes the form

$$\frac{dy}{dx} = f(y),$$

whence $$x = \int \frac{dy}{f(y)} + c.$$

If it cannot be solved for p, let it be solved for y, thus

$$y = \phi(p);$$

differentiating, $$p = \phi'(p)\frac{dp}{dx}$$

in which the variables x and p can be separated. For example, given

$$y = p^2 + \tfrac{2}{3}p^3; \qquad (1)$$

differentiating, $$p = (2p + 2p^2)\frac{dp}{dx},$$

$$dx = (2 + 2p)\,dp,$$

and integrating, $$x = 2p + p^2 + C,$$

which can be written in the form

$$(p + 1)^2 = x + c. \qquad (2)$$

Substituting the value of p derived from equation (2), namely

$$p = -1 + \sqrt{(x + c)},$$

in equation (1), we obtain

$$y = \tfrac{1}{3} - x - c + \tfrac{2}{3}(x + c)^{\frac{3}{2}},$$

or rationalizing,

$$(x + y + c - \tfrac{1}{3})^2 = \tfrac{4}{9}(x + c)^3.$$

36. Next, suppose the equation not to contain y; if it can be solved for p, it will take the directly integrable form

$$\frac{dy}{dx} = f(x);$$

if not, let it be solved for x. Then

$$x = \phi(p),$$

differentiating with reference to y,

$$\frac{1}{p} = \phi'(p)\frac{dp}{dy},$$

in which the variables y and p can be separated.

Clairaut's Equation.

37. The equation

$$y = xp + f(p), \tag{1}$$

where f denotes any function, is known as Clairaut's equation. Its complete integral is obviously

$$y = xc + f(c); \tag{2}$$

for the result of differentiating this equation with reference to x is

$$p = c,$$

which substituted in (2) produces equation (1).

The complete integral (2) ,in this case, represents a system of straight lines.

38. The conditions for equal roots in equation (1) and in equation (2) are obviously identical. Since the multiple roots of an equation satisfy the derived equation, the condition for the existence of equal roots is found by eliminating the unknown quantity between the given and derived equations. Differentiating (2) with reference to c, we have

$$0 = x + f'(c), \tag{3}$$

hence the required condition is the result of eliminating c between equation (2) and (3). The process is the same as that of finding the envelop of the system of lines (2), [see Diff. Calc., Art. 367] and the result is the singular solution of equation (1).

For example, let the differential equation be

$$y = px + \frac{a}{p}, \tag{1}$$

the complete integral is

$$y = cx + \frac{a}{c}. \tag{2}$$

Differentiating with reference to c,

$$0 = x - \frac{a}{c^2};$$

whence substituting $c = \sqrt{\frac{a}{x}}$ in equation (2), and reducing, we have

$$y^2 = 4ax \tag{3}$$

the singular solution, which is the equation of a parabola, to which the lines represented by the complete integral (2) are the tangents.

Extension of Clairaut's Equation.

39. The equation

$$y = x\,\phi(p) + f(p) \tag{1}$$

is known as *the extension of Clairaut's equation.* The relation between x and p, found as in Art. 31, is

$$p = \phi(p) + x\,\phi'(p)\frac{dp}{dx} + f'(p)\frac{dp}{dx},$$

or

$$\frac{dx}{dp} = \frac{\phi'(p)}{p - \phi(p)}\,x + \frac{f'(p)}{p - \phi(p)}.$$

This is a linear equation for x regarded as a function of p, and by means of its integral p is to be eliminated from equation (1).

Transformation of Differential Equations.

40. It is sometimes possible to reduce an equation to an integrable form by replacing one of the variables, for example y, by a new variable z, which may be a function of y, or of both x and y. As an illustration, let us take the equation

$$y^2 + y^2\left(\frac{dy}{dx}\right)^2 = 2nx + 2ny\frac{dy}{dx}.$$

Since $d(y^2) = 2y\,dy$, the equation is obviously simplified by putting $y^2 = z$, which gives

$$4z + \left(\frac{dz}{dx}\right)^2 = 8nx + 4n\frac{dz}{dx},$$

whence

$$\left(\frac{dz}{dx} - 2n\right)^2 = 4n^2 - 4z + 8nx.$$

Again, putting $v = z - 2nx$, this becomes

$$\left(\frac{dv}{dx}\right)^2 = 4(n^2 - v),$$

whence

$$\frac{dv}{2\sqrt{(n^2 - v)}} = dx.$$

Integrating, $\sqrt{(n^2 - v)} = x - c,$

or $n^2 - y^2 + 2nx = (x - c)^2.$

41. The homogeneous equation

$$xp^2 - 2yp + 4x = 0$$

may be reduced to Clairaut's form by putting

$$z = x^2, \qquad \text{whence} \qquad dx = \frac{dz}{2\sqrt{z}},$$

and $p = \frac{dy}{dx} = 2\sqrt{z}\,\frac{dy}{az} = 2\sqrt{z}p'.$

Substituting,

$$4z^{\frac{3}{2}}p'^2 - 4yz^{\frac{1}{2}}p' + 4z^{\frac{1}{2}} = 0,$$

or $y = zp + \frac{1}{p},$

and the integral is

$$y = cz + \frac{1}{c},$$

or, putting $C = -\frac{1}{c}$, $\quad x^2 + Cy + C^2 = 0.$

Trajectories.

42. A curve which cuts a system of curves at a constant angle is called a *trajectory* of the system. The case usually considered is that of the *orthogonal trajectory*, which cuts the given system at right angles. The differential equation of this trajectory is readily derived from the differential equation of the given system; for, at every point of intersection, x and y have identical values for the curve and trajectory, but, if ϕ and ϕ' denote their inclinations to the axis of x,

$$\phi = \phi' \pm \frac{\pi}{2},$$

whence $p = \tan\phi = -\cot\phi' = -\frac{1}{p'}.$

If, therefore, we substitute $-\frac{1}{p}$ for p in the differential equation of the given system of curves, the result will be the differential equation of the trajectory. The complete integral of this equation represents a system of curves each of which is an orthogonal trajectory of the given system. It is obvious, also, that the curves of the given system are the orthogonal trajectories of the new system.

43. As an illustration, *let it be required to determine the orthogonal trajectory of the circles which pass through two given points.*

Taking the straight line which passes through the given points as the axis of y, and the middle point as origin, and denoting the distance between the points by $2b$, the equation of the system of circles is

$$x^2 + y^2 + cx - b^2 = 0,$$

in which c is an arbitrary parameter. Putting this in the form

$$\frac{x^2 + y^2 - b^2}{x} + c = 0,$$

in order that c shall disappear on differentiating, we have

$$x^2 - y^2 + b^2 + 2xyp = 0$$

for the differential equation of the given system of curves. Substituting $-\frac{dx}{dy}$ for p, we obtain

$$2xydx - (x^2 - y^2 + b^2)\, dy = 0,$$

the differential equation of the trajectories. Putting $z = x^2$, this becomes

$$y\,dz - z\,dy + (y^2 - b^2)\, dy = 0,$$

of which $\frac{1}{y^2}$ is obviously an integrating factor. Thus

$$\frac{y\,dz - z\,dy}{y^2} + \left(1 - \frac{b^2}{y^2}\right) dy = 0,$$

and integrating,

$$\frac{z}{y} + y + \frac{b^2}{y} + C = 0.$$

or

$$x^2 + y^2 + b^2 + Cy = 0.$$

Hence the trajectories form a system of circles of which the axis of x is the common radical axis, and the given points are the vanishing points.

III.

Equations of the Second and Higher Orders.

44. It is shown in Art. 4. that two arbitrary constants must occur in the complete integral of a differential equation of the second order. The direct integration of an equation of the second order produces an equation of the first order containing one arbitrary constant. Thus the equation

$$\frac{d^2 y}{dx^2} + y = 0, \tag{1}$$

when multiplied by $2p$, where $p = \frac{dy}{dx}$. whence $\frac{d^2 y}{dx^2} = \frac{dp}{dx}$ becomes

$$2p\frac{dp}{dx} + 2y\frac{dy}{dx} = 0,$$

which, by direct integration, gives

$$p^2 + y^2 = c^2, \tag{2}$$

in which it is convenient to denote the constant of integration, which must obviously be positive, by c^2.

Now solving equation (2), which is of the first order, we have

$$\frac{dy}{\sqrt{(c^2 - y^2)}} = dx,$$

whence $$\sin^{-1}\frac{y}{c} = x + a,$$

or $$y = c \sin (x + a), \tag{3}$$

where x is a second constant of integration.

The complete integral (3) may also be written in the form

$$y = c \cos a \sin x + c \sin a \cos x,$$

or putting

$$A = c \cos a, \qquad B = c \sin a,$$

$$y = A \sin x + B \cos x, \tag{4}$$

in which, since A and B may have any values whatever assigned to them, they may be taken as the two independent constants of integration.

The First Integrals.

45. The equation of the next lower order resulting from the direct integration of a given differential equation is called a *first integral* of the equation. Thus equation (2) in the last article is a first integral of equation (1). The complete integral (3), in this case, is the complete integral of (2) when a is regarded as arbitrary, and the latter equation may be derived from it by differentiating and eliminating a.

If, on the other hand, we differentiate (3) and eliminate c, the result is

$$p = y \cot (x + a), \tag{5}$$

a relation between x, y, p and a, from which equation (1) may be reproduced by differentiating and eliminating a. Equation (5) is therefore also a first integral of the given equation (1); and we thus see that a differential equation of the second order has two first integrals corresponding to the two constants which appear in the complete integral.

46. Let us now consider the form (4) of the complete integral in this example. Differentiating, we have

$$p = A \cos x - B \sin x.$$

Eliminating B from (4) by means of this equation, we have

$$y \sin x + p \cos x = A. \tag{6}$$

In like manner, eliminating A, we have

$$y \cos x - p \sin x = B. \tag{7}$$

Equations (6) and (7) are the two first integrals of

$$\frac{d^2y}{dx^2} + y = 0$$

corresponding to the form (4) of its complete integral.

47. It is obvious that the two first integrals constitute two relations between x, y, p and the two constants, and therefore the elimination of p between them will produce the complete integral, or relation between x, y, and the constants. Thus, equations (2) and (5) give equation (3), and equations (6) and (7) give equation (4); and in fact any two first integrals containing independent constants give rise to a form of the complete integral.

Integrating Factors.

48. A differential equation of the second order is said to be *exact*, when it is the direct result of differentiating an equation of the first order. If any first integral of a given equation be written in such a form that the arbitrary constant disappears by differentiation, the result of its differentiation will be an exact equation, and will be equivalent to the given equation, multiplied, in general, by an integrating factor. Thus the differentiation of equation (6) in the last article gives

$$\cos x \left(y + \frac{d^2y}{dx^2} \right) = 0;$$

and equation (7) gives in like manner the integrating factor $\sin x$. Again, the solution in Art. 44 shows that p is an integrating factor of the same equation.

Geometrical Meaning of a Differential Equation of the Second Order.

If, in a differential equation of the second order, we regard x and y as the rectangular coordinates of a point, p will determine the direction in which the point is moving. We may in the equation assume any values we chose for x, y and p; the equation then determines the value of $\frac{d^2y}{dx^2}$, and this, in connection with the value of p, determines the curvature of the path in which the point moves. The equation of of the path described is a particular integral; and the complete integral represents the doubly infinite system of curves thus determined. In other words, the equation of the system contains two arbitrary constants, whose values, for the particular integral considered, may be determined by the conditions that the curve shall pass through the assumed initial point, and that at that point p shall have the assumed initial value.

50. In the example considered in the foregoing articles, the complete integral (3), Art. 44,

$$y = c \sin (x + a)$$

represents all the sinusoids whose axes coincide with the axis of x, and of which 2π is the length of a complete branch.

If c is regarded as fixed, and a alone as arbitrary, the equation represents the system of equal sinusoids which is produced by moving the sinusoid

$$y = c \sin x$$

in the direction of the axis of x; and the corresponding first integral

$$p^2 + y^2 = c^2$$

is the differential equation of this system. If we now suppose c to vary, we have the infinite number of systems of sinusoids, of different amplitudes, which constitute the doubly infinite system represented by the complete integral.

In like manner, the doubly infinite system of curves may be derived from the singly infinite system represented by any first integral, by the variation of the constant which appears in that first integral.

51. The first integral considered in the last article has the singular solution

$$y^2 = c^2,$$

representing the straight lines $y = \pm c$, the envelopes of the system of equal sinusoids; this, however, is *not* a solution of the given differential equation of the second order

$$\frac{d^2y}{dx^2}+y=0,$$

but corresponds to the integrating factor p by means of which the first integral in question was derived in Art. 44. Thus a singular solution of a first integral is not a solution of the given equation of the second order, but results simply from the mode in which the doubly infinite system of curves is decomposed into a system of systems of curves. The other first integrals of this equation which have been given in the preceding articles have no singular solutions, the singly infinite system of curves represented being in each case a system of curves intersecting only in fixed points.

Singular Solutions of Equations of the Second Order.

52. If the given differential equation of the second order be of the first degree, and we put

$$\frac{d^2y}{dx^2}=q,$$

the equation will always determine a single value of q corresponding to any assumed values of x, y and p, Thus there will be one and only one curve of the system passing through a given point in a given direction; and a point moving in such a manner as to give the proper value to q necessarily moves in one of the curves representing particular integrals. But if the equation is of the second degree, there are two real or imaginary values of q corresponding to the assumed values of x, y and p; and, when q is real, there are two curves passing through the assumed point in the assumed direction, and therefore touching one another. These two curves will in general have different curvatures, and a point moving in such a manner as to satisfy the differential equation when passing through the given point in the given direction will necessarily move in one of these two curves.

53. Now suppose the assumed values of x, y and p to be such that q has equal roots, and that the point moves in such a manner that the condition for equal roots continues to be satisfied; then it is possible that a point so moving shall at each instant have a value of q identical with the equal values of q corresponding to its position and direction of motion; just as, in the case of a differential equation of the first order, a point moving in the envelope has at each point a value of p identical with the equal values of p corresponding to that point. The curve thus described is said to *osculate* with each of the particular integral curves which touches it, that is, to have at the point of contact the same curvature with it. The equation of an osculating curve is a singular solution of the differential equation, since the simultaneous values of x y, p and q satisfy the equation; and the condition that q shall have equal roots is, in general, a relation between x, y and p, that is, a differential equation of the first order which is satisfied by the osculating curves if any exist.

54. To illustrate, let us suppose the system of curves represented by the complete integral to consist of all the circles whose centres are on the curve whose equation is

$$F(x, y) = 0. \tag{1}$$

Then if we assume any point and direction, the centres of the particular integral circles which pass through the assumed point in the assumed direction may be found by drawing a straight line through the assumed point perpendicular to the assumed direction to meet the curve represented by equation (1).

Let us suppose that this line generally meets the curve in two points, so that there are two values of q; then the case of equal roots occurs when the line is tangent to the curve. Hence, a point moving so as to satisfy the condition for equal roots describes a curve whose normals are tangent to the given curve. Such a curve is an *involute* of the given curve (1), and the circles of the system which touch it are its circles of curvature. Now it is evident that a point moving in the involute has the same values of x, y, p and q, as if it were moving in the circle of curvature at the point of contact, so that the equation of an involute satisfies the differential equation, and is therefore a singular solution. Thus the condition for equal roots in this case is the differential equation of the first order representing all the involutes of the given curve (1), and the equations of these involutes are singular solutions of the given differential equation.

Derivation of the Complete Integral from two First Integrals.

55. The relation between the complete integral and any two independent first integrals, explained in Art. 47, sometimes leads to a determination of the former without further integration.

For example, as an obvious extension of the results arrived at in Art. 48, we see that the equation

$$\frac{d^2y}{dx^2} + a^2y = 0$$

has the integrating factors

$$\cos ax, \qquad \text{and} \qquad \sin ax;$$

and since these expressions involve x only, they are also integrating factors of the more general equation

$$\frac{d^2x}{dx^2} + a^2y = X, \tag{1}$$

in which X denotes any function of x. Thus, we have

$$\cos ax \frac{d^2y}{dx^2} + a^2y \cos ax = X \cos ax,$$

an exact differential equation whose integral is

$$\cos ax \frac{dy}{dx} + a\,y \sin ax = \int X \cos ax\, dx + C. \tag{2}$$

In like manner, the integrating factor $\sin ax$ gives

$$\sin ax \frac{dy}{dx} - a\,y \cos ax = \int X \sin ax\, dx + C'. \tag{3}$$

Eliminating p between the first integrals (2) and (3), we have

$$ay = \sin ax \int X \cos ax\, dx - \cos ax \int X \sin ax\, dx$$
$$+ C \sin ax - C' \cos ax.$$

Exact Differential Equations of the Second Order.

56. An exact differential of the second order is the differential of an expression involving x, y, and p. Hence, if a given expression is exact, the derivative $\frac{d^2y}{dx^2}$ or $\frac{dp}{dx}$, being introduced by the differentiation, can occur only in the first degree; and the terms containing p in the integral may be at once derived by integrating with reference to p the terms containing $\frac{dp}{dx}$ in the given expression. Having found these terms, their complete derivative must be substracted from the given expression of the second order, and the remainder, which will be a differential expression of the first order must be examined. If this remainder multiplied by dx forms an exact differential, the sum of its integral and the terms already found is evidently an expression whose differential is the given expression. The latter is therefore, in this event, an exact differential, and the sum just mentioned is its integral.

57. For example, let the equation

$$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0 \qquad (1)$$

be given. The first member may be written

$$xy\frac{dp}{dx}+xp^2-yp.$$

Integrating the first term with respect to p, we have

$$xyp,$$

of which the complete derivative is

$$xy\frac{dp}{dx}+xp^2+yp.$$

Subtracting this from the given expression, the remainder is

$$-2yp,$$

which multiplied by dx is the exact differential $-2ydy$. Hence. equation (1) is exact, and

$$xyp-y^2=C$$

is a first integral. Separating the variables, and integrating again we have the complete integral.

$$C+y^2=C'x^2.$$

Equation (1) may also be integrated by dividing by xyp which gives

$$\frac{dp}{p}+\frac{dy}{y}-\frac{dx}{x}=0,$$

in which the three variables are separated.

Equations in which y and one or more of its Lower Derivatives do not occur.

58. As mentioned in Art. 31, an equation of the second order which does not contain y may be regarded as an equation of the first order between p and x. In fact, the method explained in that article consists in forming an equation of the second order of which the given equation is a first integral, obtaining another first integral of this equation, and then deriving the complete integral by the elimination of p.

When the given equation of the second order, not containing y, is of the first degree with respect to both the derivatives, it may be written in the form

$$\frac{d^2y}{dx^2}+f(x)\frac{dy}{dx}=\phi(x)$$

or
$$\frac{dp}{dx}+Pp=Q,$$

which is a linear equation for p. For example, the equation
$$(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}+ax=0$$

takes the linear form
$$\frac{dp}{dx}+\frac{xp}{1+x^2}=-\frac{ax}{1+x^2},$$

of which the integral is
$$p=-a+\frac{C}{\sqrt{(1+x^2)}}.$$

Integrating again,
$$y=C'-ax+C\log[x+\sqrt{(1+x^2)}]$$

59. In general. an equation of the nth order in which y does not occur and the lowest derivative is of the rth order is equivalent to an equation of the $(n-r)$th order for the determination of this derivative. For example
$$\frac{d^4y}{dx^4}=a^2\frac{d^2y}{dx^2}$$

is equivalent to
$$\frac{d^2q}{dx^2}=a^2q,$$

of which the complete integral is
$$q=ce^{ax}+c'e^{-ax}.$$

Integrating twice, we have
$$y=Ae^{ax}+Be^{-ax}+Cx+D.$$

Equations in which x does not occur.

60. An equation of the second order in which x does not occur may be reduced to an equation of the first order between y and p by the substitution
$$\frac{d^2y}{dx^2}=\frac{dp}{dx}=p\frac{dp}{dy}.$$